

# Strategy bifurcation and spatial inhomogeneity in a simple model of competing sellers

L. MITCHELL<sup>(a)</sup> and G. J. ACKLAND<sup>(b)</sup>

*SUPA, School of Physics, The University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom*

PACS 89.65.Gh – Economics; econophysics, financial markets, business and management

PACS 89.75.-k – Complex systems

PACS 89.75.Fb – Structures and organization in complex systems

**Abstract.** - We present a simple one-parameter model for spatially localised evolving agents competing for spatially localised resources. The model considers selling agents able to evolve their pricing strategy in competition for a fixed market. Despite its simplicity, the model displays extraordinarily rich behaviour. In addition to “cheap” sellers pricing to cover their costs, “expensive” sellers spontaneously appear to exploit short-term favourable situations. These expensive sellers “speciate” into discrete price bands. As well as variety in pricing strategy, the “cheap” sellers evolve a strongly correlated spatial structure, which in turn creates niches for their expensive competitors. Thus an entire ecosystem of coexisting, discrete, symmetry-breaking strategies arises.

Many economic models of marketplace interactions have been formulated (e.g., [1–3]). Generally, these systems assume complete information, and no transaction costs. That is, for interacting buyers and sellers, buyers may always search the entire space of sellers (possibly with some search cost) in order to find the best deal.

When competition is between sellers, such games generally have a stable, zero-profit (Nash) equilibrium, possibly with multiple prices [1,3]. Despite a substantial literature on spatial extensions to classic game theoretic models such as the Prisoner’s Dilemma [4–7], few models exist for simple marketplaces in which buyers cannot access the entire seller space (some exist for real-world situations, see, e.g., [8]). Again in the context of the Prisoner’s Dilemma, much progress has been made by considering players which are only adaptive by random mutation with selection [9,10] as in ecological models [11, 12]. However, this has been only infrequently applied to marketplace games, one model is described in [13].

In this paper we present a simple spatial model, similar in spirit to the Minority Game [14, 15], with limited interaction distances and random mutations with selection. The model is formulated in terms of active, evolving sellers competing for passive buyers. A dual ecological model

involves different species competing for a scarce resource.

We attempt to make the simplest possible model for a spatially distributed market with localised information and evolving price strategy. We consider a system of  $2N$  interacting agents: agents are split into one of two types, there are  $N$  selling agents (sellers) and  $N$  buying agents (buyers). Agents are placed on a 1-dimensional chain where each site contains a seller, and each link a buyer. Buyers are connected to their nearest neighbours, i.e., they have knowledge of 2 sellers (fig. 1. Each seller has capital  $C_i$  and an unvarying price  $P_i$ . Initial prices are drawn from  $P_i \in [1, P_{max}]$ .

Each iteration proceeds as follows:

1. All sellers’ capital is reduced by 2, the cost of producing enough stock for both possible buyers.
2. Each buyer visits the cheapest connected seller.
3. For each buyer visiting seller  $i$ ,  $C_i$  increases by  $P_i$ .
4. All sellers with  $C_i < 0$  are bankrupt: site  $i$  becomes vacant.
5. Vacant sites are repopulated with probability  $\gamma$ .
6. New sellers at site  $i$  have  $C_i = 0$ .

<sup>(a)</sup>E-mail: lawrence.mitchell@ed.ac.uk

<sup>(b)</sup>E-mail: g.j.ackland@ed.ac.uk

7. New sellers at site  $i$  take the price of an existing seller at randomly chosen site  $j$ ,  $P_i = P_j + dp$  ( $dp \in [\max(-\Delta, 1 - P_j), \Delta]$ ).

Note that buyers are always present, but unlike in the other games mentioned above, sellers' sites might not participate in all rounds of the game (if  $\gamma \neq 1$ ). This allows for local variation in the spatial structure and the availability of supply. Sellers are assumed to know their overhead cost (2), and will not charge below this. Stock is assumed to be perishable and thus, any unsold stock is destroyed<sup>1</sup>. The new sellers may be regarded either as independent sellers adopting their strategy from successful rivals, or franchises of those rivals.

Similarly, it is a matter of definition whether the sellers are in any sense “intelligent”. A seller makes no price adjustment between its initial appearance and bankruptcy, so in this sense exhibits no intelligence. It may be assumed to have no information about its neighbours’ strategy for the upcoming round, which would in turn prevent it from deducing an optimal strategy: as we shall see, in the evolved state there is a high turnover of shops such that two neighbours seldom compete for more than one round. The *sites*, by contrast, do have a degree of intelligence, since when their strategy is observed to have made a loss they change it to one which has been successful elsewhere. There is strong evidence that independent businesses do indeed adopt known successful business plans, or that “best practice” within a franchise spreads from one location to another.

There are three parameters in the model,  $P_{max}$ ,  $\Delta$ , and  $\gamma$ .  $P_{max}$  is simply a boundary on the initial conditions; as we shall see, for reasonable values, the mutation step  $\Delta$  affects only the timescale of reaching equilibrium:  $\gamma$  is the only parameter which governs system behaviour.

We will show that this model produces very complex behaviour, with a range of discrete but non-symmetric strategies emerging. Before doing so, we discuss what would be expected from a mean field approach.

The classic analysis for this type of demand limited competition [16] suggests that prices will be driven down to the “Bertrand equilibrium”, a level that recoups the production cost, here  $P_0 = 1$ . With the current model, there is insufficient demand to support all sellers at this price, thus there will be dead sites whose number may be estimated.

Initially, consider the case where the system is already in the Bertrand equilibrium: let the price of each seller be chosen randomly from a uniform distribution  $P \in [1, 1+\delta]$ , with small variation:  $\delta \sim \Delta \ll 1$ . In order to survive a round, each seller must sell all its stock.

Consider a live seller, at the beginning of a round it will be in one of three situations:

1. Both neighbouring sellers are dead.

<sup>1</sup>Alternately, stock could have negligible value compared to fixed costs

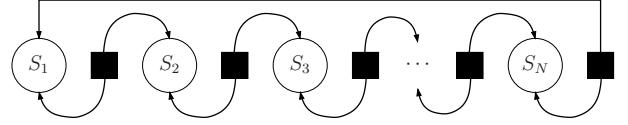


Fig. 1: Diagram of buyer-seller connections in 1D. Buyers and sellers are shown by black squares and open circles respectively. Arrows indicate the sellers an individual buyer may visit

2. One neighbouring seller is dead, while the other is alive.
3. Both neighbouring sellers are alive.

Let  $\alpha$  be the proportion of live sellers at the beginning of the round, then we can write the probability of each of the three cases as:  $p_1 = (1-\alpha)^2$ ,  $p_2 = 2\alpha(1-\alpha)$ , and  $p_3 = \alpha^2$ . In order to survive, the seller must either be in situation (1), or in situation (2) or (3) and outcompeting the live sellers. This gives a survival probability (given the uniform price distribution) of  $p_s = p_1 + \frac{p_2}{2} + p_3 \int_0^1 (1-x)^2 dx = \frac{\alpha^2}{3} - \alpha + 1$ . The proportion surviving is hence  $f(\alpha) = \alpha(\alpha^2/3 - \alpha + 1)$ . Thus, with  $\gamma = \frac{1}{2}$ , at the beginning of the next round, the proportion of live sites is  $\frac{1}{2}(1 + f(\alpha))$ ; in the steady state, this must be equal to  $\alpha$ . Solving numerically gives the proportion of live sites in the steady state as  $\alpha_{ss} \approx 0.66$ .

An alternative assumption is to search for Nash equilibrium of the game. Although our agents are constrained to have fixed price (so-called pure strategies) it is known that the ensemble average at a Nash equilibrium of pure strategies is the same as the time-average for mixed strategies, provided the pure agents do not know which strategy they are playing against [17]. Thus we might guess that out distribution of prices will resemble the mixed Nash equilibrium for the non-spatial game.

In this analysis, we assume that there is a distribution of prices  $f(p)$  which includes the Bertrand price  $P_0$ . In Nash equilibrium with mixed strategies, all strategies have the same payoff - since  $P_0$  has zero payoff, other strategies which offer zero payoff are included. Since all prices below  $P_0$  always lose, we need consider only higher prices:

$$\int_{p_i}^{\infty} (p_i - 1)f(p)dp = \int_1^{p_i} f(p)dp \quad (1)$$

whence  $f(p) = 1/p^2$ . This approach ignores the possibility that sites are unoccupied. We may include this in the analysis by allowing an unoccupied site, paying no overhead, to be part of the strategy (it has the same payoff as  $P_0$ ). It turns out, however, that the mixed strategy equilibrium does not contain this particular pure strategy: let the probability that a site plays be  $\eta$ , now suppose that an opposing site chooses to play with probability  $\beta$ . In order to maximize our expected profit, we should now choose  $\eta > \beta$  (cashing in when our opponent plays dead). Equally, however, our opponent should choose  $\beta > \eta$ , to

maximize her expected profit. Thus, the equilibrium situation is for both players to choose  $\eta = \beta = 1$ , i.e., to play every round.

Simulated results with initial prices seeded close to the Bertrand equilibrium show that the mean field assumption is invalid. We find  $\alpha = 0.71 \pm 0.01$  in the steady state, which does not agree with the prediction for  $\alpha_{ss}$ . Closer examination of the structure of the steady state in simulation shows that there is a high degree of correlation in placement of sellers. If the steady-state were a mean field, we would expect  $p(n) \approx \alpha_{ss} \forall n$ . As can be seen in fig. 3, this is evidently not the case. An ordered array of “supercheap” sellers on alternate sites forms with prices very close to  $P_0$ :  $P_i - P_0 \ll \Delta$ . The presence of such an array is stable against intrusion in the intermediate sites, as a putative new seller opening there must be cheaper than *both* neighbours, and *both* their eventual replacements to survive.

The fully correlated case, where every other seller is supercheap, has  $\alpha_{ss} = 0.75$ , while the uncorrelated case has  $\alpha_{ss} \approx 0.66$ . In between the two extremes is the actual situation. With random initial conditions, many correlated regions develop at the same time. In order for them to match at their boundaries, they must nucleate in phase, otherwise they form an antiphase boundary which cannot be removed by the addition or removal of a single supercheap seller (fig. 3). Thus the Bertrand “equilibrium” is locally stable to small perturbations, although some spatial structure is already visible (fig. 3).

Simulation of the model with a wider range of initial prices shows that the global steady-state is a good deal more complex: a range of high-price sellers coexist with the cheap ones: fig. 2. These sellers exploit temporary monopoly situations where adjacent sites are dead.

This extraordinary behaviour is at variance with a conventional demand-limited picture, and can be likened to biological speciation. Several distinct seller types emerge, which cannot mutate into one another. The expensive sellers need not have an infinite lifetime: because of the replicator dynamics it suffices that each should be replicated once in its average lifetime.

Due to the highly correlated environment which expensive sellers occupy, a mean field analysis is insufficient. Instead, consider the first band ( $P = 2$ ) of expensive sellers in 1D: these survive if they sell to, on average, one buyer per turn. The possible changes in the capital  $C$  of such an expensive seller, assuming its neighbours are cheap, are:

1.  $\Delta C = P - 1$  if both neighbouring sellers are dead,
2.  $\Delta C = P/2 - 1$  if one neighbour is dead and the other is alive,
3.  $\Delta C = -1$  if both neighbours are alive.

$C$  therefore carries out a random walk halting when the capital becomes negative. The naïve guess is that this walk is biased in favour of the negative step; one might

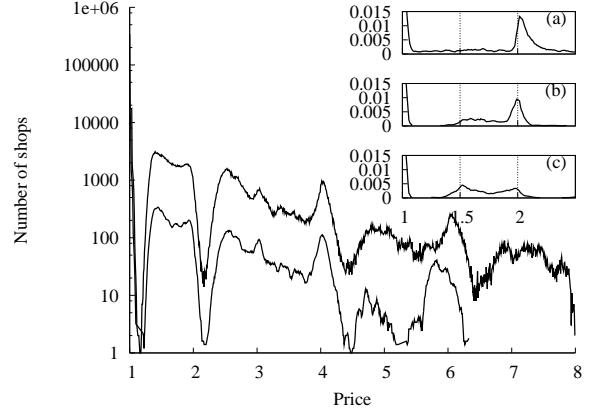


Fig. 2: Steady state price distribution for  $N = 10^5$  and  $N = 10^6$ ,  $\gamma = \frac{1}{2}$ ,  $P_{max} = 8$ ,  $\Delta = 0.04$ , price shown in units of  $P_0$ , y axis is absolute number of shops. Insets shows evolution of the  $P = 2$  band to a steady state for  $N = 10^5$ , y axis is fraction of total shops. Note that main figure is a log-linear plot, while insets are linear. (a) is after 100 timesteps, (b) after 500, and (c) in the steady state. A sharp band forms initially above  $P = 2$  which creates a niche for shops with prices  $P < 2$ , the band then migrates downward and broadens to that in (c). Data binned by rounding to 3 decimal places, and subsequently smoothed with a 5 point average. The main features of the graph are size independent and reproducible (as shown), and sharpen with reduced  $\Delta$

expect that  $p_1 = (1 - \alpha)^2$ ,  $p_2 = 2\alpha(1 - \alpha)$ , and  $p_3 = \alpha^2$  with  $\alpha \approx 0.68$ . However, simulation tells us that the mean lifetime of expensive sellers scales with the lifetime of the game, our naïve guess must therefore be incorrect.

It turns out that the long-lived expensive sellers occupy favourable niches: the “supercheap” sites of the correlated phase, i.e., their second neighbours are supercheap. In the limiting case, this means that their first neighbour competitors are dead 50% of the time. This changes the step direction bias in the random walk described above since the probabilities primarily depend on  $\gamma$  rather than  $\alpha$ : the mean lifetime of such a walk with  $\gamma \leq \frac{1}{2}$  is infinite.

These niches would appear to allow arbitrarily high prices, and any seller charging  $\geq 2$  to survive. However, one can apply the ideas of Bertrand competition to the expensive sellers: on a long enough timescale they will set up adjacent to one another, and capital will be transferred to the cheaper seller.

The discrete trading rounds mean that integer prices will have better short term survival prospects: e.g., a sale to one buyer at 4 in the first trading period will ensure survival for two rounds, while 3.9 will only survive one. In this scenario with two potential buyers, the advantage for odd integer price is less: e.g., a price of 3 has to sell twice to survive an extra round compared with 2. If one starts with a homogenous distribution of initial prices, this leads to “speciation”: symmetry breaking in the preferred price band favouring marginally above integer value, fig. 2. Remarkably, once the speciation has occurred, the character

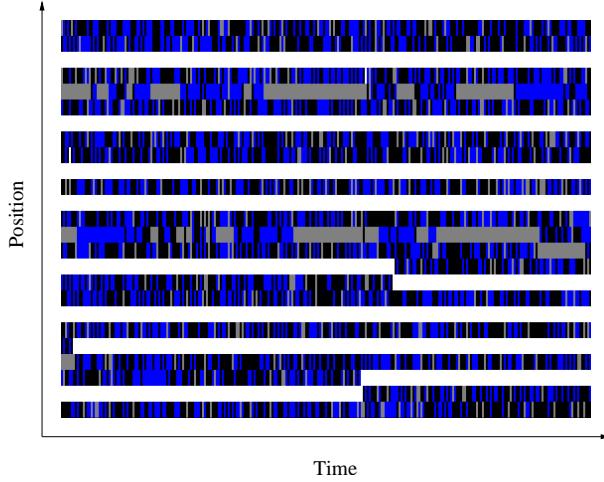


Fig. 3: Migration of phase boundaries and correlation amongst cheap shops over time. Each row represents one site (after rebirth), coloured according to price. A dead site is black; a site with  $P \in [1, 1.00004]$  is white; a site with  $P \in [1.00004, 1.008]$  is grey; a site with  $P \in [1.008, 1.28]$  is blue. Each column represents one timestep, the picture shows around 300 timesteps in total

of the competition changes again. ‘‘Intraspecies’’ competition between sellers in the same price band becomes critical, and prices below the integer values become viable, until a balance is reached between intra- and inter-species competition.

Although this analysis requires that trading rounds be discrete, the main features (heavily favoured prices) are still present if trading happens in a stochastic manner. For stochastic dynamics, a buyer is chosen at random to go shopping, and a seller is chosen at random to pay an overhead, this is repeated such that the expected number of times a buyer goes shopping is one, this completes one trading round. The bankruptcy and rebirth dynamics proceed as before. Since buyers may now visit a seller more than once, there is no upper bound on the amount of stock a seller may sell, we therefore set the quantity of stock to  $\infty$ , and thus the marginal cost to zero. Despite this, prices at integer multiples of  $P = 1$  are still favoured (fig. 4).

We may remove a further restriction on the original model by not requiring that sellers charge above the marginal cost. In this case, favoured prices still exist, and do so for a range of connectivities: for a seller with  $d$  potential buyers (paying an overhead  $d$ ), prices of  $P = dn/m$ ,  $n, m \in \mathbb{Z}^+$  are favoured (fig. 4 inset), corresponding to attracting  $m/n$  buyers on average per round.

We now consider whether such expensive sellers are in some sense beneficial. Due to the existence of dead sites and the limited interaction distance, demand is not completely satisfied. The introduction of a wider range of prices results in both a larger total population and more demand being satisfied.

The dependence of the lifetimes of the expensive sellers

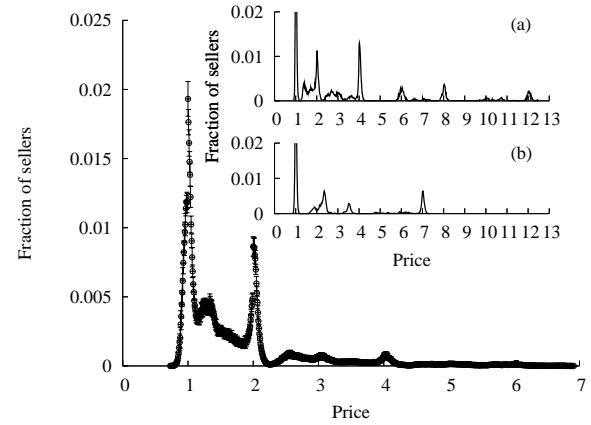


Fig. 4: Steady state price distribution for stochastic dynamics with  $N = 4 \times 10^4$ ,  $\gamma = \frac{1}{2}$ ,  $\Delta = 0.04$ . Distribution averaged over final state of 20 ensembles, error bars show standard error in mean. Inset shows steady state price distribution for discrete trading rounds and no minimum price for connectivities of (a) 4, and (b) 7 buyers per seller (overhead of 4 and 7 respectively), the peaks at 4 and 7 in (a) and (b) correspond to sellers attracting one buyer

on  $\gamma$  tells us that this parameter may be used to characterize the distribution. In the limit  $\gamma \rightarrow 1$  all sellers charge the Bertrand price, as there is never any chance of expensive sellers being the only option for consumers. Equally,  $\gamma = 0$  leads to an essentially random distribution of sellers (depending on initial conditions). In between these two extremes, we expect some kind of transition from a regime with expensive sellers to one without around  $\gamma = \frac{1}{2}$ : if  $\gamma < \frac{1}{2}$  the random walk of the expensive sellers is biased in favour of the upward, profit-making step, allowing them to survive indefinitely.

By monitoring fluctuations in a simulation we can see that the system undergoes a transition at a critical value of  $\gamma \approx \frac{1}{2}$  in which the variance in the number of live sellers diverges (fig. 5).

We see further evidence of a transition when considering the mean unsatisfied demand, being approximately constant for  $\gamma < \frac{1}{2}$  and quadratic for  $\gamma > \frac{1}{2}$  (fig. 6). If there were no transition, we would expect the demand to be quadratic in  $\gamma$  for all values: this is indeed the case if we force the system into a Bertrand steady state by specifying initial conditions accordingly.

The system has a metastable regime for  $\gamma < \frac{1}{2}$ . If the initial conditions only sample the Bertrand regime ( $P \approx 1$ ), then it remains in such a steady state indefinitely. Equally, if the initial conditions sample the whole price spectrum, then the final steady state contains multiple price bands. In order for the system to escape from the Bertrand state, it requires a nucleation of expensive sellers which cannot happen through mutations in prices (due to adverse selective pressures on intermediate prices). Equally, for small system sizes, fluctuations may eliminate high price bands which cannot be repopulated.

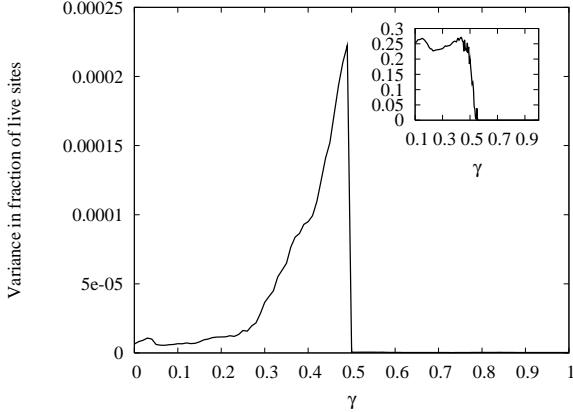


Fig. 5: Variance in the time-averaged mean of the fraction of live sellers (before rebirth) as a function of  $\gamma$ ,  $N = 10^4$ , smoothed using 5 point average separately above and below  $\gamma = 0.5$ . Inset shows the fraction of expensive shops (after rebirth) as a function of  $\gamma$

According to Nash [18], it is possible for multiple strategies to coexist provided that no individual can do any better by changing their strategy. In our system, changes in price at a site are discontinuous, as are changes in the competing strategies of the neighbours (each changes only when the shop's capital goes to zero). These discrete, localised changes prevent the system finding a perfect Nash equilibrium. To apply the notion of a Nash equilibrium one has to assume that the evolution of prices is equivalent to the sites behaving as rational agents: it is possible that this is the case only in the infinite time limit, not reached by our simulations. Another hypothesis about evolving, replicating systems is that the system as a whole organises to maximise the number of replicators (here, sellers) [19]. In fig. 6 we see good evidence for this: the expensive sellers become viable when they are able to increase the total number of living sellers above the Bertrand solution. A side effect of this is to minimise the unsatisfied demand.

We have shown that the obvious generalization of a classic Bertrand-Edgeworth game has some surprising results. The classical Bertrand equilibrium is not necessarily reached, as the system is able to self-organise to produce niches where different strategies can flourish. Further, we have shown that random mutation and selection can (in the case of restricted initial conditions) produce the expected Nash result. That is, sellers need not be active in selecting strategies, the selective force against badly performing members is enough to bring the system to equilibrium.

The spontaneous production of evolutionary niches in an initially homogeneous space has strong parallels in evolutionary ecology. We can envisage a similar situation where the “sellers” become individuals foraging for food. The “cheap sellers” represent foragers which are efficient at finding the food, but have a high metabolic rate and need to feed often. The “expensive sellers” are less effi-

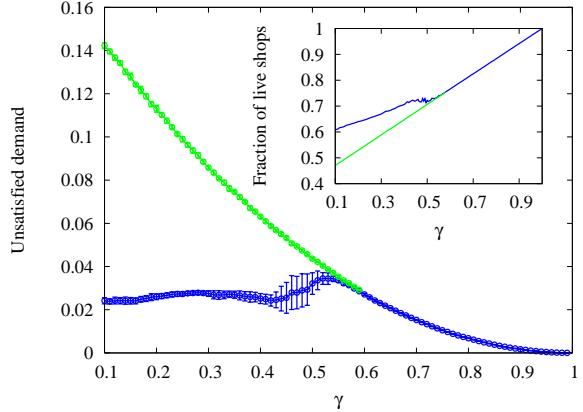


Fig. 6: Time-averaged unsatisfied demand as a function of the birth rate,  $\gamma$ , in enforced Bertrand steady state (green) and multiple price steady state (blue),  $N = 10^4$ , mean taken as a time average over  $2 \times 10^4$  timesteps after the steady state is reached. Inset shows corresponding fraction of live sites after rebirth for Bertrand (green) and non-Bertrand (blue) steady state. Error bars show standard error in the mean

cient at foraging, but can survive for longer on the same amount of food.

\* \* \*

This work was produced by the NANIA collaboration ([www.ph.ed.ac.uk/nania](http://www.ph.ed.ac.uk/nania)) funded by EPSRC grant T11753.

## REFERENCES

- [1] VARIAN H. R., *The American Economic Review*, **70** (1980) 651
- [2] SALOP S. and STIGLITZ J. E., *Review of Economic Studies*, **44** (1977) 493
- [3] SALOP S. and STIGLITZ J. E., *The American Economic Review*, **72** (1982) 1121
- [4] NOWAK M. A. and MAY R. M., *Nature*, **359** (1992) 826
- [5] HUBERMAN B. and GLANCE N., *PNAS*, **90** (1993) 7716
- [6] FELDMAN B. and NAGEL K., *Lectures in Complex Systems, Papers from the summer school held in Santa Fe, NM, USA, 1992*, edited by STEIN D. and NADEL L., Vol. **5** (Addison-Wesley) 1993, pp. 603-614
- [7] NOWAK M. A. and SIGMUND K., *Science*, **303** (2004) 793
- [8] KIRMAN A. P. and VRIEND N. J., *Journal of Economic Dynamics & Control*, **25** (2001) 459
- [9] AXELROD R., *Genetic Algorithms and Simulated Annealing*, edited by DAVIS L. (Morgan Kaufman, Los Altos, CA) 1987, pp. 32-41
- [10] AXELROD R. and DION D., *Science*, **242** (1988) 1385
- [11] HOLLAND J. H., *Adaptation in Natural and Artificial Systems: an introductory analysis with applications to biology, control, and artificial intelligence* (MIT Press, Cambridge, MA) 1992
- [12] DIECKMANN U., LAW R. and METZ J. A. J. (Editors), *The Geometry Of Ecological Interactions: Simplifying*

- Spatial Complexity* (Cambridge University Press, Cambridge) 2000
- [13] NAGEL K., SHUBIK M., PACZUSKI M. and BAK P., *Physica A*, **287** (2000) 546
  - [14] CHALLET D. and ZHANG Y.-C., *Physica A*, **246** (1997) 407
  - [15] CHOE S. C., JOHNSON N. F. and HUI P. M., *Physical Review E*, **70** (2004) 055101(R)
  - [16] BERTRAND J., *Journal des Savants*, (1883) 499
  - [17] SMITH J. M., *Evolution and the Theory of Games* (Cambridge University Press, Cambridge) 1982
  - [18] NASH J. F., *The Annals of Mathematics*, **54** (1951) 286
  - [19] ACKLAND G. J., *Journal of Theoretical Biology*, **227** (2004) 121